ELSEVIER

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

J. Math. Anal. Appl. 286 (2003) 652–663

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Uncertainty principles for Jacobi expansions ☆

Zhongkai Li * and Limin Liu

Department of Mathematics, Capital Normal University, Beijing 100037, China

Received 6 December 2002

Submitted by R.H. Torres

Abstract

In this paper an uncertainty principle for Jacobi expansions is derived, as a generalization of that for ultraspherical expansions by Rösler and Voit. Indeed a stronger inequality is proved, which is new even for Fourier cosine or ultraspherical expansions. A complex base of exponential type on the torus $\{z \in \mathbb{C}\colon |z| = 1\}$ related to Jacobi polynomials is introduced, which are the eigenfunctions both of certain differential–difference operators of the first order and the second order. An uncertainty principle related to such exponential base is also proved.

Keywords: Uncertainty principle; Jacobi series; Differential–difference operator

1. Introduction and main results

We will prove an uncertainty principle for Jacobi expansions, as a generalization of that for ultraspherical expansions by Rösler and Voit in [11]. Indeed a stronger inequality will be proved, which is new even for Fourier cosine or ultraspherical expansions (see (3), (6) and (7) below). To describe these, we first recall some facts about the Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$, which are normalized so that $R_n^{(\alpha,\beta)}(1) = 1$. The Jacobi polynomials arise from the generating function [14, (4.4.5)]

$$\sum_{n=0}^{\infty} \binom{n+\alpha}{n} R_n^{(\alpha,\beta)}(x)\omega^n = 2^{\alpha+\beta} A^{-1}(1-\omega+A)^{-\alpha}(1+\omega+A)^{-\beta}$$

☆ The work is supported by the National Natural Science Foundation of China and by the Natural Science Foundation of Beijing.

* Corresponding author.
E-mail address: lizk@mail.cnu.edu.cn (Zh. Li).

0022-247X/$ – see front matter
doi:10.1016/S0022-247X(03)00507-9

with $A = (1 - 2x\omega + \omega^2)^{1/2}$, and satisfy the recurrence relation [14, (4.5.1)]

$$
\begin{aligned}
&2(n+\alpha+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)R_{n+1}^{(\alpha,\beta)}(x) \\
&\quad = (2n+\alpha+\beta+1)\bigl\{(2n+\alpha+\beta+2)(2n+\alpha+\beta)x+\alpha^2-\beta^2\bigr\}R_n^{(\alpha,\beta)}(x) \\
&\qquad - 2n(n+\beta)(2n+\alpha+\beta+2)R_{n-1}^{(\alpha,\beta)}(x)
\end{aligned}
$$

for $n \geqslant 1$, with $R_0^{(\alpha,\beta)}(x) = 1$ and $R_{-1}^{(\alpha,\beta)}(x) = 0$. The ultraspherical polynomials are the Jacobi polynomials when $\alpha = \beta$.

For $\alpha, \beta > -1$, we denote by $L^2([a,b], d\mu_{\alpha\beta})$ the Hilbert space with the inner product $\langle f, g\rangle = \int_a^b f\bar{g}\,d\mu_{\alpha\beta}$ and the norm $\|f\|_{2,[a,b]} = \langle f, f\rangle^{1/2}$, where $[a,b] = [0,\pi]$ or $[-\pi,\pi]$ and $d\mu_{\alpha\beta}(t) = 2^{\alpha+\beta+1}|\sin t/2|^{2\alpha+1}|\cos t/2|^{2\beta+1}\,dt$. It is well known that $\{\phi_n(t) = R_n^{(\alpha,\beta)}(\cos t)\}_{n\geqslant 0}$ is a complete orthogonal set in $L^2([0,\pi], d\mu_{\alpha\beta})$, and satisfies the following differential equation (cf. [6, (2.5)], [14, (4.2.1)]):

$$L_{\alpha\beta}\phi_n = -n(n+\alpha+\beta+1)\phi_n, \tag{1}$$

where

$$L_{\alpha\beta} = \frac{d^2}{dt^2} + \frac{(\alpha+\beta+1)\cos t + \alpha - \beta}{\sin t}\frac{d}{dt}.$$

The Jacobi expansion of a function $f \in L^2([0,\pi], d\mu_{\alpha\beta})$ is defined by

$$f(t) \sim \sum_{n=0}^{\infty} \hat{f}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos t), \tag{2}$$

where $\omega_n^{(\alpha,\beta)} = \|R_n^{(\alpha,\beta)}\|_{2,[0,\pi]}^{-2}$ and $\hat{f}(n) = \int_0^\pi f(t)R_n^{(\alpha,\beta)}(\cos t)\,d\mu_{\alpha\beta}(t)$.

The main result of the present paper is to prove

Theorem 1. *For* $f \in L^2([0,\pi], d\mu_{\alpha\beta})$,

$$\int_0^\pi \bigl|f(t)\bigr|^2 \sin^2 t\,d\mu_{\alpha\beta}(t)\cdot \operatorname{var}_{\alpha\beta}(f) \geqslant (\alpha+1)^2\bigl|\tau_{\alpha\beta}(f)\bigr|^2, \tag{3}$$

where the generalized mean $\tau_{\alpha\beta}$ *is defined by*

$$\tau_{\alpha\beta}(f) = \int_0^\pi R_1^{(\alpha,\beta)}(\cos t)\cdot\bigl|f(t)\bigr|^2 d\mu_{\alpha\beta}(t) \tag{4}$$

and the frequency variance $\operatorname{var}_{\alpha\beta}(f)$ *by*

$$\operatorname{var}_{\alpha\beta}(f) = \sum_{n=0}^{\infty} n(n+\alpha+\beta+1)\omega_n^{(\alpha,\beta)}\bigl|\hat{f}(n)\bigr|^2 \in [0,\infty]. \tag{5}$$

The constant $(\alpha+1)^2$ *is optimal.*

From inequality (3), we have the uncertainty principle as

Corollary 2. *For* $f \in L^2([0,\pi], d\mu_{\alpha\beta})$ *with* $\|f\|_{2,[0,\pi]} = 1$,

$$\bigl(1-\tau_{\alpha\beta}(f)\bigr)\left(\frac{\beta+1}{\alpha+1}+\tau_{\alpha\beta}(f)\right)\operatorname{var}_{\alpha\beta}(f) \geqslant \frac{1}{4}(\alpha+\beta+2)^2\bigl|\tau_{\alpha\beta}(f)\bigr|^2. \tag{6}$$

The constant $(\alpha+\beta+2)^2/4$ *is optimal.*

When $\alpha=\beta$, (6) is identical with the uncertainty relation of ultraspherical expansions in [11], that is

$$\bigl(1-\bigl|\tau_{\alpha\alpha}(f)\bigr|^2\bigr)\operatorname{var}_{\alpha\alpha}(f) \geqslant (\alpha+1)^2\bigl|\tau_{\alpha\alpha}(f)\bigr|^2. \tag{7}$$

The above inequality (7) obtained in [11] was motivated by the localization-frequency version of Heisenberg–Weyl uncertainty principle for the torus $\{e^{it} \in \mathbb{C}\colon t \in (-\pi,\pi]\}$ first introduced and discussed in [1,2], and its generalization to the unit sphere $S^2 \subset \mathbb{R}^3$ in [10]. Indeed, a spherical version on $S^d \subset \mathbb{R}^{d+1}$ is derived in [11] as a consequence of (7). For further uncertainty principles on S^d and ultraspherical expansions, see [3,13].

As in [11], to prove the theorem and the corollary, we need to factorize the second order differential operator $L_{\alpha\beta}$ by introducing a differential–difference operator T on the doubled interval $[-\pi,\pi]$. Such operator T unifies the properties of the Jacobi polynomials $R_n^{(\alpha,\beta)}(\cos t)$ and their conjugate polynomials $R_{n-1}^{(\alpha+1,\beta+1)}(\cos t)\sin t$, which make it possible to define the complex polynomial base $E_n(t)$ of exponential type. These have strong similarities to $\cos nt$, $\sin nt$ and e^{int} in Fourier series. The details about these will be contained in Section 2. We note that the differential–difference operator and the complex base of exponential type are different from those defined and studied by Dunkl (see [4,5,8]). The proof of Theorem 1 and its corollary is given in Section 3, and an uncertainty principle in terms of the complex polynomial base $E_n(t)$ is proved in Section 4. It is worthwhile to note that in definition (4) of $\tau_{\alpha\beta}(f)$, we use $R_1^{(\alpha,\beta)}(\cos t)$ in place of $\cos t$ in the case of ultraspherical expansion (see [11, (2.8)]). Indeed, from [14, (4.5.1)],

$$R_1^{(\alpha,\beta)}(\cos t) = \frac{(\alpha+\beta+2)\cos t+\alpha-\beta}{2\alpha+2}, \tag{8}$$

and $R_1^{(\alpha,\alpha)}(\cos t) = \cos t$.

2. Differential–difference operator and complex base

For a function $f(t)$ with expansion (2), according to [6], its conjugate Jacobi series is defined by

$$\tilde f(t) \sim \sum_{n=1}^{\infty} \hat f(n)\frac{n\omega_n^{(\alpha,\beta)}}{2\alpha+2} R_{n-1}^{(\alpha+1,\beta+1)}(\cos t)\sin t,$$

based on which, the H^p theory for Jacobi expansions was established in [7]. For $\alpha=\beta$, the related harmonic analysis was studied in [9] in great details. The functions $\phi_n(t) = R_n^{(\alpha,\beta)}(\cos t)$ and

$$\tilde{\phi}_n(t) = \frac{n}{2\alpha+2} R_{n-1}^{(\alpha+1,\beta+1)}(\cos t)\sin t$$

play the role of $\cos nt$ and $\sin nt$. But unlike the classical case, the functions $\tilde{\phi}_n(t)$ satisfy an equation significantly different from (1) of $\phi_n(t)$, which is

$$\tilde{L}_{\alpha\beta}\tilde{\phi}_n = -n(n+\alpha+\beta+1)\tilde{\phi}_n, \tag{9}$$

where

$$\tilde{L}_{\alpha\beta} = L_{\alpha\beta} - \frac{\alpha+\beta+1+(\alpha-\beta)\cos t}{\sin^2 t}.$$

To unify the expressions of Eqs. (1) and (9), we will introduce a first order differential–difference operator T for the functions defined on the doubled interval $[-\pi,\pi]$ as

$$Tf(t) = \frac{d}{dt}f(t) + \frac{(\alpha+\beta+1)\cos t+\alpha-\beta}{2\sin t}\big(f(t)-f(-t)\big). \tag{10}$$

Then direct computing leads to

$$T^2 f(t) = L_{\alpha\beta}f(t) - \frac{1}{2}\frac{\alpha+\beta+1+(\alpha-\beta)\cos t}{\sin^2 t}\big(f(t)-f(-t)\big). \tag{11}$$

It is clear that the operator T supplies a factorization of the second order differential operator $L_{\alpha\beta}$, namely $L_{\alpha\beta}f(t) = T^2 f(t)$ for even functions $f(t)$ on $[-\pi,\pi]$. Moreover we have

Lemma 3. *For $t \in [-\pi,\pi]$,*

$$T^2\phi_n(t) = -\lambda_n^2\phi_n(t), \qquad T^2\tilde{\phi}_n(t) = -\lambda_n^2\tilde{\phi}_n(t),$$

and

$$T\phi_n(t) = -(n+\alpha+\beta+1)\tilde{\phi}_n(t), \qquad T\tilde{\phi}_n(t) = n\phi_n(t)$$

with $\lambda_n = \sqrt{n(n+\alpha+\beta+1)}$.

Proof. The first pair of equations are just (1) and (9) from the evenness of ϕ_n and $\tilde{\phi}_n$. And from [6, (2.9)] or [14, (4.21.7)],

$$T\phi_n(t) = \phi_n'(t) = -(n+\alpha+\beta+1)\tilde{\phi}_n(t),$$

and from what have been proved,

$$T\tilde{\phi}_n(t) = -\frac{T(T\phi_n(t))}{n+\alpha+\beta+1} = \frac{\lambda_n^2}{n+\alpha+\beta+1}\phi_n(t) = n\phi_n(t),$$

and thus the second pair of equations follows. □

Motivated by Lemma 3 and the classical relation $e^{int} = \cos nt + i\sin nt$, we introduce the complex base functions $E_n(t)$ of exponential type in terms of $\phi_n(t)$ and $\tilde{\phi}_n(t)$ by $E_0 = 1/\sqrt{2}$, and for $n \geqslant 1$,

$$
\begin{aligned}
E_n(t) &= \frac{1}{2}\left\{\phi_n(t) + i\sqrt{\frac{n+\alpha+\beta+1}{n}}\tilde{\phi}_n(t)\right\} \\
&= \frac{1}{2}\left\{R_n^{(\alpha,\beta)}(\cos t) + i\,\frac{\lambda_n}{2\alpha+2}R_{n-1}^{(\alpha+1,\beta+1)}(\cos t)\sin t\right\}, \qquad (12) \\
E_{-n}(t) &= \overline{E_n(t)}. \qquad (13)
\end{aligned}
$$

Lemma 4. *The system* $\{E_n(t)\colon\ n = 0, \pm 1, \pm 2, \ldots\}$ *is an orthogonal set in* $L^2([-\pi,\pi], d\mu_{\alpha\beta})$ *and* $\int_{-\pi}^{\pi} |E_n(t)|^2\, d\mu_{\alpha\beta}(t) = \omega_{|n|}^{(\alpha,\beta)\,-1}$.

Proof. The orthogonality follows from that of $\{\phi_n(t)\}$ and $\{\tilde{\phi}_n(t)\}$ and that $\int_{-\pi}^{\pi} \phi_n\tilde{\phi}_m\, d\mu_{\alpha\beta} = 0$ for any m, n, and the equality from [6, (2.2)]. □

We remark that functions $E_n(t)$ defined by (12) and (13) are different from the complex base introduced and studied by Dunkl in [4,5] (cf. [8] also), moreover the differential–difference operators T and T^2 are also different from those defined by him. Dunkl's study was based on the h-harmonicity and h-analyticity on the unit disc related to a finite reflection group. The advantage of the form of $E_n(t)$ here is that they are the eigenfunctions both of T and T^2, which resemble the functions e^{int} as $(d/dt)e^{\pm int} = \pm ine^{\pm int}$ and $(d^2/dt^2)e^{\pm int} = -n^2e^{\pm int}$. We state this as

Lemma 5. *For* $n \geqslant 0$, $TE_{\pm n}(t) = \pm i\lambda_n E_{\pm n}(t)$, $T^2E_{\pm n}(t) = -\lambda_n^2 E_{\pm n}(t)$.

Proof. The proof follows from Lemma 3 directly. □

For f defined on $[-\pi,\pi]$, the expansion of f in terms of $\{E_n(t)\}_{n=-\infty}^{\infty}$ is defined by

$$
f(t) \sim \sum_{n=-\infty}^{\infty} c_n(f)\omega_{|n|}^{(\alpha,\beta)}E_n(t), \qquad (14)
$$

where $c_n(f) = \langle f, E_n\rangle = \int_{-\pi}^{\pi} f(t)\overline{E_n(t)}\, d\mu_{\alpha\beta}(t)$. One can also define the expansion of f in terms of $\{\phi_n(t), \tilde{\phi}_n(t)\}$ as

$$
f(t) \sim a_0(f)\omega_0^{(\alpha,\beta)} + \sum_{n=1}^{\infty}\omega_n^{(\alpha,\beta)}\left\{a_n(f)\phi_n(t) + b_n(f)\frac{n+\alpha+\beta+1}{n}\tilde{\phi}_n(t)\right\},
$$

where $a_n(f) = (1/2)\int_{-\pi}^{\pi} f(t)\phi_n(t)\, d\mu_{\alpha\beta}(t)$, $b_n(f) = (1/2)\int_{-\pi}^{\pi} f(t)\tilde{\phi}_n(t)\, d\mu_{\alpha\beta}(t)$. It is obvious that $c_0(f) = \sqrt{2}a_0(f)$ and for $n \geqslant 1$,

$$
c_{\pm n}(f) = a_n(f) \mp i\sqrt{\frac{n+\alpha+\beta+1}{n}}b_n(f).
$$

Proposition 6. *For* $f \in C^1([-\pi,\pi])$, $c_n(Tf) = i(\operatorname{sgn} n)\lambda_{|n|}c_n(f)$,

$$
a_n(Tf) = (n+\alpha+\beta+1)b_n(f), \qquad b_n(Tf) = -na_n(f),
$$

and for $f \in C^2([-\pi,\pi])$, $c_n(T^2 f) = -\lambda_{|n|}^2 c_n(f)$, $a_n(T^2 f) = -\lambda_n^2 a_n(f)$, *and* $b_n(T^2 f) = -\lambda_n^2 b_n(f)$.

The proposition is an easy consequence of Lemmas 3, 5 and the following lemma.

Lemma 7. T *is anti-selfadjoint, that is, for* $f, g \in C^1[-\pi,\pi]$,

$$\int_{-\pi}^{\pi} Tf(t)\cdot\overline{g(t)}\,d\mu_{\alpha\beta}(t) = -\int_{-\pi}^{\pi} f(t)\cdot\overline{Tg(t)}\,d\mu_{\alpha\beta}(t).$$

Proof. For $f, g \in C^1[-\pi,\pi]$, using (10) and taking partial integration yields

$$\begin{aligned}
&\int_{-\pi}^{\pi} Tf(t)\overline{g(t)}\,d\mu_{\alpha\beta}(t)\\
&\quad= -\int_{-\pi}^{\pi} f(t)\overline{g'(t)}\,d\mu_{\alpha\beta}(t) - \int_{-\pi}^{\pi} f(t)\overline{g(t)}\frac{(\alpha+\beta+1)\cos t+\alpha-\beta}{\sin t}\,d\mu_{\alpha\beta}(t)\\
&\qquad+\int_{-\pi}^{\pi}\frac{(\alpha+\beta+1)\cos t+\alpha-\beta}{2\sin t}\big(f(t)-f(-t)\big)\overline{g(t)}\,d\mu_{\alpha\beta}(t),
\end{aligned}$$

the later two terms being identical with

$$-\int_{-\pi}^{\pi} f(t)\frac{(\alpha+\beta+1)\cos t+\alpha-\beta}{2\sin t}\big(\overline{g(t)-g(-t)}\big)\,d\mu_{\alpha\beta}(t).$$

Thus the lemma is proved. □

3. Proof of Theorem 1 and Corollary 2

First let $f \in C^2([0,\pi])$. To use the anti-selfadjointness of the operator T, we extend $f(t)$ to $[-\pi,\pi]$ by $f(-t) = f(t)$. Then from (5), Proposition 6, Parseval's formula and Lemma 7,

$$\begin{aligned}
\operatorname{var}_{\alpha\beta}(f) &= -\int_0^{\pi} L_{\alpha\beta} f(t)\cdot\overline{f(t)}\,d\mu_{\alpha\beta}(t) = -\frac{1}{2}\int_{-\pi}^{\pi} T^2 f(t)\cdot\overline{f(t)}\,d\mu_{\alpha\beta}(t)\\
&= \frac{1}{2}\int_{-\pi}^{\pi}\big|Tf(t)\big|^2\,d\mu_{\alpha\beta}(t),
\end{aligned}\tag{15}$$

and from (10), $T(|f|^2) = (d/dt)|f(t)|^2 = \bar f f' + \bar f' f = 2\operatorname{Re}(\bar f f') = 2\operatorname{Re}(\bar f Tf)$. By the Cauchy–Schwartz inequality, we have

$$\int_{-\pi}^{\pi} \left|f(t)\right|^2 \sin^2 t \, d\mu_{\alpha\beta}(t) \cdot \int_{-\pi}^{\pi} \left|Tf(t)\right|^2 d\,\mu_{\alpha\beta}(t) \geqslant \left\{ \int_{-\pi}^{\pi} |\sin t| \left|\overline{f(t)}Tf(t)\right| d\mu_{\alpha\beta}(t) \right\}^2$$
$$\geqslant \left| \int_{-\pi}^{\pi} \sin t \cdot \mathrm{Re}\left\{\overline{f(t)}Tf(t)\right\} d\mu_{\alpha\beta}(t) \right|^2 = \frac{1}{4} \left| \int_{-\pi}^{\pi} \sin t \cdot T\left(|f|^2\right)(t)\, d\mu_{\alpha\beta}(t) \right|^2. \tag{16}$$

Now using the anti-selfadjointness of the operator T and noting that $T(\sin t) = (2\alpha + 2) \times R_1^{(\alpha,\beta)}(\cos t)$, then in view of (4), the last expression of (16) becomes

$$\frac{1}{4} \left| \int_{-\pi}^{\pi} T(\sin t) \left|f(t)\right|^2 d\mu_{\alpha\beta}(t) \right|^2 = 4(\alpha+1)^2 \left|\tau_{\alpha\beta}(f)\right|^2.$$

Thus (3) follows from (15) and (16) immediately.

To obtain (6) from (3), it is sufficient to show that for $\|f\|_{2,[0,\pi]} = 1$,

$$\frac{(\alpha+\beta+2)^2}{4(\alpha+1)^2} \int_{0}^{\pi} \left|f(t)\right|^2 \sin^2 t \, d\mu_{\alpha\beta}(t) \leqslant \left(1 - \tau_{\alpha\beta}(f)\right) \left(\frac{\beta+1}{\alpha+1} + \tau_{\alpha\beta}(f) \right). \tag{17}$$

Indeed, from (4) and (8), the right-hand side of (17) is

$$\frac{\beta+1}{\alpha+1} + \frac{\alpha-\beta}{\alpha+1}\tau_{\alpha\beta}(f) - \tau_{\alpha\beta}(f)^2$$
$$= \frac{\beta+1}{\alpha+1} + \frac{\alpha-\beta}{\alpha+1} \left\{ \frac{\alpha+\beta+2}{2(\alpha+1)} \int_{0}^{\pi} \cos t \cdot \left|f(t)\right|^2 d\mu_{\alpha\beta}(t) + \frac{\alpha-\beta}{2(\alpha+1)} \right\}$$
$$- \left\{ \frac{\alpha+\beta+2}{2(\alpha+1)} \int_{0}^{\pi} \cos t \cdot \left|f(t)\right|^2 d\mu_{\alpha\beta}(t) + \frac{\alpha-\beta}{2(\alpha+1)} \right\}^2$$
$$= \frac{(\alpha+\beta+2)^2}{4(\alpha+1)^2} \left\{ 1 - \left(\int_{0}^{\pi} \cos t \cdot \left|f(t)\right|^2 d\mu_{\alpha\beta}(t) \right)^2 \right\}.$$

Again using the Cauchy–Schwartz inequality on the second term in the brackets above and $\|f\|_{2,[0,\pi]} = 1$, the last expression is bounded from below by

$$\frac{(\alpha+\beta+2)^2}{4(\alpha+1)^2} \left\{ 1 - \int_{0}^{\pi} \cos^2 t \cdot \left|f(t)\right|^2 d\mu_{\alpha\beta}(t) \right\}$$
$$= \frac{(\alpha+\beta+2)^2}{4(\alpha+1)^2} \int_{0}^{\pi} \left|f(t)\right|^2 \sin^2 t \, d\mu_{\alpha\beta}(t),$$

which proves (17), and then (6) follows.

The validity of (3) and (6) for general $f \in L^2([0,\pi], d\mu_{\alpha\beta})$ with a finite value $\mathrm{var}_{\alpha\beta}(f)$ of (5) follows from the density on L^2 of the set of polynomials (cf. [11, p. 631]).

Finally we prove the optimality of the constants in (3) and (6). In view of (17), we only need to show this for (6). For this purpose, following the procedure of [11], we consider the (α,β)-densities $g_\epsilon(t)$ of Gaussian measures on $[0,\pi]$ as $\epsilon \downarrow 0$, defined by

$$g_\epsilon(t) = c_\epsilon \sum_{n=0}^{\infty} e^{-\epsilon\lambda_n^2} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos t), \tag{18}$$

where the constant c_ϵ is such that $\|g_\epsilon\|_{2,[0,\pi]} = 1$. It is easy to see that the optimality of (6) is a straight consequence of the following lemma.

Lemma 8. *For $\alpha, \beta > -1$, we have*

(i) $$\lim_{\epsilon\to 0+} \epsilon \cdot \mathrm{var}_{\alpha\beta}(g_\epsilon) = \frac{\alpha+1}{2};$$

(ii) $$\lim_{\epsilon\to 0+} \frac{1}{\epsilon}\bigl(1 - \tau_{\alpha\beta}(g_\epsilon)\bigr) = \frac{\alpha+\beta+2}{2}.$$

Proof. We will use the fact that for a fixed A and for $r > -1$,

$$\lim_{\epsilon\to 0+} \epsilon^{(r+1)/2} \sum_{n=1}^{\infty} n^r e^{-\epsilon n(n+A)} = \frac{1}{2}\Gamma\left(\frac{r+1}{2}\right). \tag{19}$$

Equation (19) is proved in [11] (Lemma 3.4 there and with $A = 2\alpha+1$), by transferring the series in (19) into an integral in terms of the relation

$$\left| \sum_{n=1}^{\infty} h(n) - \int_0^{\infty} h(x)\,dx \right| \leqslant c_r \epsilon^{-\max\{r,0\}/2} \tag{20}$$

with $h(x) = x^r e^{-\epsilon x(x+A)}$. In [11], the factor $\epsilon^{-r/2}$ for $r > 0$ was missed and the fact that the monotonicity intervals of $h(x)$ depends upon the parameter ϵ was neglected. It is easy to find that $h(x)$ is increasing on $[0, x_\epsilon]$ and decreasing on $[x_\epsilon, \infty)$ with $x_\epsilon = [\sqrt{A^2\epsilon^2 + 8r\epsilon} - A\epsilon]/(4\epsilon)$. Now by splitting the integral interval $[0,\infty)$ into the union of $[n-1, n)$ and using the monotonicity of $h(x)$, we have

$$\begin{aligned}
\int_0^1 h(x)\,dx - \int_{[x_\epsilon]}^{[x_\epsilon]+1} h(x)\,dx &\leqslant \sum_{n=1}^{\infty} h(n) - \int_1^{\infty} h(x)\,dx \\
&\leqslant h\bigl([x_\epsilon]\bigr) + h\bigl([x_\epsilon]+1\bigr) - \int_{[x_\epsilon]}^{[x_\epsilon]+1} h(x)\,dx.
\end{aligned}$$

Consequently (20) follows, since $h(x) \simeq \epsilon^{-r/2}$ for $x \in [[x_\epsilon], [x_\epsilon]+1]$. That (20) implies (19) is an easy exercise.

Now we turn to the proof of (i) and (ii). It follows from (5) and (18) that

$$\mathrm{var}_{\alpha\beta}(g_\epsilon) = c_\epsilon^2 \sum_{n=0}^{\infty} \lambda_n^2 \omega_n^{(\alpha,\beta)} e^{-2\epsilon\lambda_n^2}, \qquad c_\epsilon^{-2} = \sum_{n=0}^{\infty} \omega_n^{(\alpha,\beta)} e^{-2\epsilon\lambda_n^2}. \tag{21}$$

Since (cf. [6, (2.2)])

$$\omega_n^{(\alpha,\beta)} = A_{\alpha\beta} n^{2\alpha+1}\bigl(1 + O(n^{-1})\bigr) \tag{22}$$

and $\lambda_n^2 \omega_n^{(\alpha,\beta)} = A_{\alpha\beta} n^{2\alpha+3}(1 + O(n^{-1}))$, in the formula of $\mathrm{var}_{\alpha\beta}(g_\epsilon)$ one needs to deal with a ratio, and multiplication with ϵ is equivalent to multiplication by $(2\epsilon)^{\alpha+2}/2$ on the numerator and $(2\epsilon)^{\alpha+1}$ in the denominator which is the right setting for (19) with $r = 2\alpha + 3$ and $r = 2\alpha + 1$, respectively, and so applying (19) yields part (i).

To prove part (ii), we rewrite the recurrence relation of $\{R_n^{(\alpha,\beta)}(x)\}$ (see Section 1) into

$$R_1^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(x) = a_n R_{n+1}^{(\alpha,\beta)}(x) + b_n R_n^{(\alpha,\beta)}(x) + c_n R_{n-1}^{(\alpha,\beta)}(x),$$

where $b_n = 1 - a_n - c_n$,

$$a_n = \frac{(\alpha+\beta+2)(n+\alpha+\beta+1)(n+\alpha+1)}{(\alpha+1)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \tag{23}$$

$$c_n = \frac{(\alpha+\beta+2)n(n+\beta)}{(\alpha+1)(2n+\alpha+\beta)(2n+\alpha+\beta+1)} \tag{24}$$

for $n \geqslant 1$, and $a_0 = 1$, $c_0 = 0$. Thus we have

$$\begin{aligned}
\tau_{\alpha\beta}(g_\epsilon) &= \int_0^\pi R_1^{(\alpha,\beta)}(\cos t) g_\epsilon(t) \cdot \overline{g_\epsilon(t)}\, d\mu_{\alpha\beta}(t) \\
&= c_\epsilon^2 \sum_{n=0}^{\infty} \omega_n^{(\alpha,\beta)} \bigl\{a_n e^{-\epsilon(\lambda_n^2+\lambda_{n+1}^2)} + b_n e^{-2\epsilon\lambda_n^2} + c_n e^{-\epsilon(\lambda_n^2+\lambda_{n-1}^2)}\bigr\} \\
&= c_\epsilon^2 \sum_{n=0}^{\infty} \omega_n^{(\alpha,\beta)} \bigl\{a_n e^{-\epsilon(2n+\alpha+\beta+2)} + b_n + c_n e^{\epsilon(2n+\alpha+\beta)}\bigr\} e^{-2\epsilon\lambda_n^2}.
\end{aligned}$$

Hence by (21),

$$1 - \tau_{\alpha\beta}(g_\epsilon) = c_\epsilon^2 \sum_{n=0}^{\infty} \omega_n^{(\alpha,\beta)} F_n(\epsilon) e^{-2\epsilon n(n+\alpha+\beta)} e^{\epsilon(\alpha+\beta)}, \tag{25}$$

where $F_n(\epsilon) = (a_n + c_n) e^{-\epsilon(2n+\alpha+\beta)} - a_n e^{-2\epsilon(2n+\alpha+\beta+1)} - c_n$. Applying the Taylor formula and (23) and (24) to $F_n(\epsilon)$ gives

$$F_n(\epsilon) = \frac{\alpha+\beta+2}{\alpha+1}\bigl[(\alpha+1)\epsilon - n^2\epsilon^2 + O(n\epsilon^2 + n^3\epsilon^3)\bigr],$$

and then substituting this into (25) and using (19), (21) and (22) yields

$$\lim_{\epsilon\to 0+} \frac{1}{\epsilon}\bigl(1 - \tau_{\alpha\beta}(g_\epsilon)\bigr) = \frac{\alpha+\beta+2}{\alpha+1}\left[\alpha + 1 - \frac{\alpha+1}{2}\right] = \frac{\alpha+\beta+2}{2},$$

which proves part (ii) of the lemma. □

Before we finish this section, we point out that inequality (3) is an improvement of (6) and (7) even for $\alpha = \beta = -1/2$. In fact if we take $f_1(t) = \sqrt{2/\pi}\cos(t/2)$, then $\|f_1\|_{2,[0,\pi]} = 1$ and $1 - |\tau_{-1/2,-1/2}(f_1)|^2 = 3/4 > 1/2 = \int_0^\pi |f_1(t)|^2 \sin^2 t\,dt$.

4. An uncertainty principle for complex base

In this section we will prove an uncertainty relation for the expansions in terms of the complex base $\{E_n(t)\}$.

For $f \in C^2([-\pi,\pi])$ with $\|f\|_{2,[-\pi,\pi]} = 1$, the $(\alpha,\beta)-$variance of f is defined by

$$\operatorname{var}^*_{\alpha\beta}(f) = \left|\langle T^2 f, f\rangle\right| - \left|\langle Tf, f\rangle\right|^2,$$

or, if f has expansion (14), equivalently by

$$\operatorname{var}^*_{\alpha\beta}(f) = \sum_{n=-\infty}^{\infty} \lambda^2_{|n|}\omega^{(\alpha,\beta)}_{|n|}\left|c_n(f)\right|^2 - \left(\sum_{n=-\infty}^{\infty}(\operatorname{sgn} n)\lambda_{|n|}\omega^{(\alpha,\beta)}_{|n|}\left|c_n(f)\right|^2\right)^2, \tag{26}$$

in view of Proposition 6. The related mean is defined by

$$\tau^*_{\alpha\beta}(f) = \int_{-\pi}^{\pi} E_1(t)\left|f(t)\right|^2 d\mu_{\alpha\beta}(t).$$

In the definition of $\tau^*_{\alpha\beta}(f)$, we use $E_1(t)$ as a replacement of e^{it} in the case of usual Fourier series. From (8) and (12),

$$E_1(t) = \frac{1}{2}\left\{\frac{(\alpha+\beta+2)\cos t + \alpha - \beta}{2\alpha+2} + i\,\frac{\sqrt{\alpha+\beta+2}}{2\alpha+2}\sin t\right\}. \tag{27}$$

Theorem 9. *For $f \in L^2([-\pi,\pi], d\mu_{\alpha\beta})$ with $\|f\|_{2,[-\pi,\pi]} = 1$, the following holds*:

$$\begin{aligned}&\left\|\left(E_1(t) - \tau^*_{\alpha\beta}(f)\right)f\right\|^2_{2,[-\pi,\pi]} \cdot \operatorname{var}^*_{\alpha\beta}(f)\\ &\quad\geqslant \frac{\alpha+\beta+2}{4}\left|\tau^*_{\alpha\beta}(f) - \gamma\tau^*_{\alpha\beta}(f_0) - \delta\|f_0\|^2_{2,[-\pi,\pi]}\right|^2,\end{aligned} \tag{28}$$

where $f_0(t) = (f(t) - f(-t))/2$, $\gamma = \frac{2(\alpha+\beta+1)}{\alpha+\beta+2}$ and $\delta = \frac{\alpha-\beta}{(2\alpha+2)(\alpha+\beta+2)}$.

Proof. First let $f \in C^2([-\pi,\pi])$. Then it is easy to find that

$$\operatorname{var}^*_{\alpha\beta}(f) = \left\|\left(T - \langle Tf, f\rangle\right)f\right\|^2_{2,[-\pi,\pi]}.$$

By the Cauchy–Schwartz inequality we have

$$\begin{aligned}&\left\|\left(E_1(t) - \tau^*_{\alpha\beta}(f)\right)f\right\|_{2,[-\pi,\pi]} \cdot \operatorname{var}^*_{\alpha\beta}(f)^{1/2}\\ &\quad\geqslant \left|\int_{-\pi}^{\pi}\left(E_1(t) - \tau^*_{\alpha\beta}(f)\right)f(t)\left(\overline{Tf(t) - \langle Tf, f\rangle f(t)}\right)d\mu_{\alpha\beta}(t)\right|\end{aligned}$$

$$= \left| \int_{-\pi}^{\pi} \bigl(E_1(t) - \tau_{\alpha\beta}^*(f)\bigr) f(t) \overline{Tf(t)}\, d\mu_{\alpha\beta} \right|, \tag{29}$$

and similarly,

$$\begin{aligned} &\bigl\| \bigl(E_1(t) - \tau_{\alpha\beta}^*(f)\bigr) f \bigr\|_{2,[-\pi,\pi]} \cdot \operatorname{var}_{\alpha\beta}^*(f)^{1/2} \\ &\quad \geqslant \left| \int_{-\pi}^{\pi} \bigl(E_1(t) - \tau_{\alpha\beta}^*(f)\bigr) \overline{f(t)}\, Tf(t)\, d\mu_{\alpha\beta}(t) \right|. \end{aligned} \tag{30}$$

Summing the both sides of (29) and (30) gives

$$\begin{aligned} &\bigl\| \bigl(E_1(t) - \tau_{\alpha\beta}^*(f)\bigr) f \bigr\|_{2,[-\pi,\pi]} \cdot \operatorname{var}_{\alpha\beta}^*(f)^{1/2} \\ &\quad \geqslant \left| \int_{-\pi}^{\pi} \bigl(E_1(t) - \tau_{\alpha\beta}^*(f)\bigr) \operatorname{Re}\bigl(f(t)\overline{Tf(t)}\bigr)\, d\mu_{\alpha\beta}(t) \right|. \end{aligned} \tag{31}$$

It is easy to see that

$$\operatorname{Re}(f\overline{Tf}) = \frac{1}{2} T\bigl(|f|^2\bigr) + \frac{(\alpha+\beta+1)\cos t + \alpha - \beta}{\sin t} |f_0|^2.$$

From Lemmas 7 and 5, we have

$$\begin{aligned} \int_{-\pi}^{\pi} \bigl(E_1(t) - \tau_{\alpha\beta}^*(f)\bigr) T\bigl(|f|^2\bigr)\, d\mu_{\alpha\beta}(t) &= -\int_{-\pi}^{\pi} TE_1(t) \cdot \bigl|f(t)\bigr|^2\, d\mu_{\alpha\beta}(t) \\ &= -i\sqrt{\alpha+\beta+2} \cdot \tau_{\alpha\beta}^*(f), \end{aligned} \tag{32}$$

and from (27) and using the fact that

$$\frac{(\alpha+\beta+1)\cos t + \alpha - \beta}{\sin t} \bigl|f_0(t)\bigr|^2$$

is an odd function, we get

$$\begin{aligned} &\int_{-\pi}^{\pi} \bigl(E_1(t) - \tau_{\alpha\beta}^*(f)\bigr) \cdot \frac{(\alpha+\beta+1)\cos t + \alpha - \beta}{\sin t} |f_0|^2\, d\mu_{\alpha\beta}(t) \\ &\quad = \frac{i\sqrt{\alpha+\beta+2}}{4(\alpha+1)} \int_{-\pi}^{\pi} \bigl[(\alpha+\beta+1)\cos t + \alpha - \beta\bigr] \bigl|f_0(t)\bigr|^2\, d\mu_{\alpha\beta}(t) \\ &\quad = \frac{i\sqrt{\alpha+\beta+2}}{2} \bigl\{ \gamma \tau_{\alpha\beta}^*(f_0) + \delta \|f_0\|_{2,[-\pi,\pi]}^2 \bigr\} \end{aligned} \tag{33}$$

with γ, δ as given in the theorem. Substituting (32) and (33) into (31) yields the desired inequality (28).

That (28) holds for general $f \in L^2([-\pi,\pi], d\mu_{\alpha\beta})$ with a finite value $\operatorname{var}_{\alpha\beta}^*(f)$ of (26) follows also from the density on L^2 of the set of polynomials. $\square$

Note that while inequality (28) looks somewhat complicated, it implies Corollary 2 by restricting f as an even function on $[-\pi, \pi]$; moreover, when $\alpha = \beta$, the right-hand side of (28) becomes

$$\frac{\alpha+1}{2}\left|\tau^*_{\alpha\alpha}(f) - \frac{2\alpha+1}{\alpha+1}\tau^*_{\alpha\alpha}(f_0)\right|^2,$$

which is quite bright (comparing this with the result on generalized Hankel transforms in [12]); and when $\alpha = \beta = -1/2$, we regain from (28) the classical uncertainty principle for the torus $(1 - |\tau(f)|^2)\operatorname{var}(f) \geqslant |\tau(f)|^2/4$ for f with $\|f\|_2 = 1$ (see [1,2,11]).

References

[1] E. Breitenberger, Uncertainty measures and uncertainty relations for angle observables, Found. Phys. 15 (1985) 353–364.

[2] P. Carruthers, M.M. Nieto, Coherent states and the number-phase uncertainty relation, Phys. Rev. Lett. 14 (1965) 387–389.

[3] D.L. Donoho, P.B. Stark, Uncertainty principles and signal recovery, SIAM J. Appl. Math. 49 (1989) 906–931.

[4] C.F. Dunkl, Differential–difference operators associated to reflection groups, Trans. Amer. Math. Soc. 311 (1989) 167–183.

[5] C.F. Dunkl, Poisson and Cauchy kernels for orthogonal polynomials with dihedral symmetry, J. Math. Anal. Appl. 143 (1989) 459–470.

[6] Zh.-K. Li, Conjugate Jacobi series and conjugate functions, J. Approx. Theory 86 (1996) 179–196.

[7] Zh.-K. Li, Hardy spaces for Jacobi expansions, Analysis 16 (1996) 27–49.

[8] L.-M. Liu, Zh.-K. Li, Harmonic analysis on extended Jacobi expansions: an application of Dunkl's theory, in: T. He, P. Shiue, Zh.-K. Li (Eds.), Analysis, Combinatorics and Computing, Nova Science, New York, 2002, pp. 319–340.

[9] B. Muckenhoupt, E.M. Stein, Classical expansions and their relation to conjugate harmonic functions, Trans. Amer. Math. Soc. 118 (1965) 17–92.

[10] F.J. Narcowich, J.D. Ward, Nonstationary wavelets on the m-sphere for scattered date, Appl. Comput. Harmon. Anal. 3 (1996) 324–336.

[11] M. Rösler, M. Voit, An uncertainty principle for ultraspherical expansions, J. Math. Anal. Appl. 209 (1997) 624–634.

[12] M. Rösler, M. Voit, An uncertainty principle for Hankel transforms, Proc. Amer. Math. Soc. 127 (1999) 183–194.

[13] R.S. Strichartz, Uncertainty principles in harmonic analysis, J. Funct. Anal. 84 (1989) 97–114.

[14] G. Szegö, Orthogonal Polynomials, in: Amer. Math. Soc. Colloq. Publ., Vol. 23, 4th ed., American Mathematical Society, Providence, RI, 1975.